

OU-HET 190  
 December, 1994  
 revised version

## Continuum Annulus Amplitudes from the Two-Matrix Model <sup>1</sup>

M. Anazawa <sup>†</sup>, A. Ishikawa <sup>‡</sup>  
 and  
 H. Itoyama <sup>†</sup>

<sup>†</sup> Department of Physics,  
 Faculty of Science, Osaka University,  
 Toyonaka, Osaka, 560 Japan  
 and

<sup>‡</sup> Uji Research Center, Yukawa Institute for Theoretical Physics,  
 Kyoto University, Uji, Kyoto, 611 Japan

### Abstract

An explicit expression for continuum annulus amplitudes having boundary lengths  $\ell_1$  and  $\ell_2$  is obtained from the two-matrix model for the case of the unitary series;  $(p, q) = (m+1, m)$ . In the limit of vanishing cosmological constant, we find an integral representation of these amplitudes which is reproduced, for the cases of the  $m = 2$  ( $c = 0$ ) and the  $m \rightarrow \infty$  ( $c = 1$ ), by a continuum approach consisting of quantum mechanics of loops and a matter system integrated over the modular parameter of the annulus. We comment on a possible relation to the unconventional branch of the Liouville gravity.

---

<sup>1</sup>This work is supported in part by Grant-in-Aid for Scientific Research (06221245) in Priority Area and (05640347) from the Ministry of Education, Japan.

One of the intriguing properties of the noncritical strings with  $c \leq 1$  is that the macroscopic  $n$ -loop amplitudes take a suggestive form in terms of the boundary lengths [1] which may inspire a geometrical interpretation. Properties of macroscopic loop amplitudes encompass those of microscopic loop amplitudes from which we directly extract the susceptibility and the operator dimensions of the continuum theory. They are, in principle, directly comparable with the results from the continuum path integrals on the geometry of annulus and the ones with more boundaries. The macroscopic loops may, in addition, represent the effects of boundary interactions of the theory. The derivation of the continuum loop (annulus) amplitudes has been given in the one-matrix model at the multicritical points both from the orthogonal polynomial approach [1] (see also [2]) and from the Schwinger-Dyson approach [3]. No comparable work has been done, on the other hand, for the case of the two-matrix model. (See [1, 4]). In this letter, we report on a progress in this direction.

The two-loop correlators we start with are

$$W_{11}(\zeta_1, \zeta_2; \mu) = \langle \langle \text{Tr} \frac{1}{X_1 - \hat{M}} \text{Tr} \frac{1}{X_2 - \hat{M}} \rangle \rangle , \quad (1)$$

$$W_{22}(\xi_1, \xi_2; \mu) = \langle \langle \text{Tr} \frac{1}{Y_1 - \hat{N}} \text{Tr} \frac{1}{Y_2 - \hat{N}} \rangle \rangle , \quad (2)$$

$$W_{12}(\zeta, \xi; \mu) = \langle \langle \text{Tr} \frac{1}{X - \hat{M}} \text{Tr} \frac{1}{Y - \hat{N}} \rangle \rangle . \quad (3)$$

Here,  $\hat{M}$  and  $\hat{N}$  are the matrix variables of the two-matrix model.  $X$ 's and  $Y$ 's are the bare boundary cosmological constants and  $\zeta$ 's and  $\xi$ 's are the renormalized boundary cosmological constants in the sense of eq. (6) below. We denote by  $\langle \langle \cdots \rangle \rangle$  the averaging in the planar limit. Formulas have been obtained of these correlators in [5] for the unitary cases  $(p, q) = (m+1, m)$ ,  $c = 1 - \frac{6}{m(m+1)}$  by finding a parametrization  $\zeta = \mu^m \cosh m\theta$ ,  $\xi = \mu^m \cosh m\tau$ ;

$$\mu \frac{\partial}{\partial \mu} W_{11}(\zeta_1, \zeta_2; \mu) = 2 \frac{\partial}{\partial \zeta_1} \frac{\partial}{\partial \zeta_2} \sum_{k=1}^{m-1} \frac{\sinh k\theta_1}{\sinh m\theta_1} \frac{\sinh k\theta_2}{\sinh m\theta_2} , \quad (4)$$

$$\mu \frac{\partial}{\partial \mu} W_{12}(\zeta, \xi; \mu) = 2 \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \xi} \sum_{k=1}^{m-1} (-)^{m-k} \frac{\sinh k\theta_1}{\sinh m\theta_1} \frac{\sinh k\theta_2}{\sinh m\theta_2} . \quad (5)$$

We denote the renormalized cosmological constant by  $\mu^{2m} \equiv M^2$  and  $W_{11}, W_{12}$  and  $W_{22}$  generically by  $W$ .<sup>2</sup> In what follows, we examine annulus amplitudes  $w(\ell_1, \ell_2)_c$  having boundary lengths  $\ell_1$  and  $\ell_2$ .

---

<sup>2</sup> For the sake of simplicity and space, we have dealt with eq. (4) explicitly in this letter. Similar expressions can be derived from eq. (5).

$W(\zeta_1, \zeta_2; \mu)$  and  $w(\ell_1, \ell_2; \mu)_c$  are related by the Laplace transform

$$\begin{aligned} W(\zeta_1, \zeta_2; \mu) &= \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 e^{-\zeta_1 \ell_1} e^{-\zeta_2 \ell_2} w(\ell_1, \ell_2; \mu)_c \\ &\equiv \mathcal{L}[w(\ell_1, \ell_2)_c]. \end{aligned} \quad (6)$$

We have found the following formula for the inverse Laplace image

$$\mathcal{L}^{-1}\left[\frac{\partial}{\partial \zeta} \frac{\sinh k\theta}{\sinh m\theta}\right] = -\frac{M\ell}{\pi} \sin \frac{k\pi}{m} K_{\frac{k}{m}}(M\ell). \quad (7)$$

Note that  $K_\nu(z)$  is the modified Bessel function. From eqs. (4),(6),(7), we find

$$\begin{aligned} \frac{\partial}{\partial M} w(\ell_1, \ell_2)_c &= \mathcal{L}^{-1}\left[\frac{\partial}{\partial M} W(\zeta_1, \zeta_2)\right] \\ &= \frac{2M\ell_1\ell_2}{m\pi^2} \sum_{k=1}^{m-1} \left(\sin \frac{k\pi}{m}\right)^2 K_{\frac{k}{m}}(M\ell_1) K_{\frac{k}{m}}(M\ell_2). \end{aligned} \quad (8)$$

Integrating once, we obtain

$$w(\ell_1, \ell_2)_c = \frac{2}{m\pi^2} \frac{M\ell_1\ell_2}{\ell_1 + \ell_2} \sum_{k=1}^{m-1} \left(\sin \frac{k\pi}{m}\right)^2 K_{\frac{k}{m}}(M\ell_1) K_{1-\frac{k}{m}}(M\ell_2). \quad (9)$$

This is our main formula whose implications will be discussed below. A similar but distinct formula is seen in [6]. An outline of the derivation of eq. (7) as well as that of eq. (9) will be given in the end of this letter.

One can easily check that eq. (9) reproduces the well-known two-loop amplitude for the case of  $(p, q) = (3, 2)$  [1, 6];

$$w(\ell_1, \ell_2)_{c=0} = \frac{1}{2\pi} \frac{\sqrt{\ell_1\ell_2}}{\ell_1 + \ell_2} e^{-M(\ell_1 + \ell_2)}. \quad (10)$$

Let us now take a close look at the limit of vanishing cosmological constant  $M \rightarrow 0$ . From the asymptotic form of the product of two modified Bessel functions, we find

$$\lim_{M \rightarrow 0} M \left(\sin \frac{k\pi}{m}\right)^2 K_{\frac{k}{m}}(M\ell_1) K_{1-\frac{k}{m}}(M\ell_2) = \left(\frac{\pi}{2}\right)^2 \frac{1}{\Gamma(\frac{k}{m})\Gamma(1-\frac{k}{m})} \left(\frac{\ell_1}{2}\right)^{-\frac{k}{m}} \left(\frac{\ell_2}{2}\right)^{\frac{k}{m}-1}. \quad (11)$$

We denote the two-loop amplitude in the limit of vanishing cosmological constant by

$$w(\ell_1, \ell_2)_c^{M \rightarrow 0} \equiv \lim_{M \rightarrow 0} w(\ell_1, \ell_2)_c. \quad (12)$$

From eq. (9) and eq. (11), we find

$$\begin{aligned} w(\ell_1, \ell_2)_c^{M \rightarrow 0} &= \frac{1}{m\pi} \frac{\ell_1}{\ell_1 + \ell_2} \sum_{k=1}^{m-1} \sin \frac{k\pi}{m} \left(\frac{\ell_2}{\ell_1}\right)^{\frac{k}{m}} \\ &= \frac{1}{m\pi} \sum_{n=0}^{\infty} \sum_{k=1}^{m-1} \sin(n + \frac{k}{m})\pi \left(\frac{\ell_2}{\ell_1}\right)^{n+\frac{k}{m}} \quad , \quad \text{for } \ell_1 > \ell_2. \end{aligned} \quad (13)$$

In the Ising ( $m = 3$ ) case, we can easily obtain a more explicit form of the two-loop amplitude from eq. (13),

$$w(\ell_1, \ell_2)_{c=1/2}^{M \rightarrow 0} = \frac{1}{2\sqrt{3}\pi} \frac{(\ell_1 \ell_2)^{1/3}}{\ell_1 + \ell_2} \left( (\ell_1)^{1/3} + (\ell_2)^{1/3} \right). \quad (14)$$

This is consistent with the result in [1].

So far, we have found an explicit expression for the continuum annulus amplitudes (eq. (9)) as well as the one in the limit of vanishing cosmological constant (eq. (13)). We now study how this limit may be reproduced by a continuum framework. ( See also [7] for a treatment at the proper-time gauge with the assumed weight multiplicity). In this limit, the functional integral measure of the matrix models concentrates on the boundaries; graphs are all degenerate and the only interaction which would take place is at the boundaries. We first point out an integral representation for  $w(\ell_1, \ell_2)_c$  for the case of vanishing cosmological constant.

$$\begin{aligned} w(\ell_1, \ell_2)_c^{M \rightarrow 0} &= \frac{\sqrt{2\beta}}{m\pi} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_2 / \ell_1)^2}{8\pi\beta t}} \\ &\quad \times \sum_{n=0}^{\infty} \sum_{k=1}^{m-1} \left(n + \frac{k}{m}\right) \sin(n + \frac{k}{m})\pi \quad e^{-2\pi\beta(n+\frac{k}{m})^2 t} \quad , \end{aligned} \quad (15)$$

where  $\beta$  is an arbitrary parameter.

In the case of pure gravity ( $m = 2$ ),

$$\begin{aligned} w(\ell_1, \ell_2)_{c=0}^{M \rightarrow 0} &= \frac{1}{4\pi} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_2 / \ell_1)^2}{8\pi t}} \\ &\quad \times \sum_{n=0}^{\infty} (-)^n (2n+1) e^{-2\pi(n+\frac{1}{2})^2 t} \quad . \end{aligned} \quad (16)$$

The Jacobi triplet product identity  $\eta(q)^3 = [q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)]^3 = - \sum_{k=-\infty}^{\infty} (-)^k k q^{1/2(k-1/2)^2}$

converts this expression into

$$\frac{1}{4\pi} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_2 / \ell_1)^2}{8\pi t}} \eta(q = e^{-4\pi t})^3 \quad . \quad (17)$$

Note that we have set  $\beta = 1$  in this expression in order to make a comparison with the continuum result written in the standard notation.

The appearance of the Dedekind function is intriguing and we pause to discuss how eq. (17) is reproduced in continuum two-dimensional gravity coupled to the conformal matter in the topology of annulus. The standard treatment at the conformal gauge yields the vacuum-to-vacuum transition amplitude

$$\mathcal{Z}_{\text{cont}} = \int_0^\infty dt \left( \frac{1}{\Omega(CKV)} \frac{\langle \psi | \frac{\partial \hat{g}}{\partial t} \rangle}{\langle \psi | \psi \rangle^{1/2}} (\det' P_1^\dagger P_1)^{1/2} \right)_{\hat{g}} Z_\phi(t) Z_m(t) \quad . \quad (18)$$

Note that the lengths  $\ell_1$  and  $\ell_2$  are not incorporated in this formula. We have chosen the reference metric  $\hat{g} = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}$ , where  $t$  is the modular parameter of the annulus.  $Z_\phi(t)$  is the contribution from the path integral with the Liouville action and  $Z_m(t)$  generically represents the contribution from conformal matter fields. The rest of the notations are standard and we leave them to review articles. ( See, for instance, [8].) After some calculation, we obtain

$$\frac{1}{\Omega(CKV)} \frac{\langle \psi | \frac{\partial \hat{g}}{\partial t} \rangle}{\langle \psi | \psi \rangle^{1/2}} = \frac{\sqrt{2}}{t} \quad (19)$$

$$\begin{aligned} \sqrt{2}(\det' P_1^\dagger P_1)^{1/2} &= (\det' \Delta_{\hat{g}})_{\text{Dirichlet}} = (\det' \Delta_{\hat{g}})_{\text{Neumann}} \\ &= 2t\eta(q = e^{-4\pi t})^2 \quad , \end{aligned} \quad (20)$$

which means that the parenthesis  $(\cdots)_{\hat{g}}$  in eq. (18) is equal to  $2\eta(q = e^{-4\pi t})^2$ . As for the  $Z_\phi$ , the argument of [9] provides a translationally invariant flat measure  $[d\phi]$  and the Liouville action which includes the zero mode  $\phi_0(\sigma^1)$ . ( We expand the Liouville field as  $\phi(\sigma^1, \sigma^2) = \phi_0(\sigma^1) + \sum_{n \neq 0} \phi_n(\sigma^1) e^{-2\pi i n \sigma^2} \equiv \phi_0(\sigma^1) + \tilde{\phi}(\sigma^1, \sigma^2)$ .) The difficulty of an attempt to find an agreement between eq. (17) and eq. (18) is that the zero mode part is not identifiable as the length operator  $\ell(\sigma^1) \equiv \int_0^1 d\sigma^2 e^{\frac{\gamma}{2}\phi(\sigma^1, \sigma^2)}$ . This can be seen, for example, in

$$\dot{\phi}_0(\sigma^1) = \frac{2}{\gamma} \frac{d}{d\sigma^1} \log \ell(\sigma^1) - \frac{\int_0^1 d\sigma^2 \dot{\tilde{\phi}}(\sigma^1, \sigma^2) e^{\frac{\gamma}{2}\tilde{\phi}(\sigma^1, \sigma^2)}}{\int_0^1 d\sigma^2 e^{\frac{\gamma}{2}\tilde{\phi}(\sigma^1, \sigma^2)}} \quad . \quad (21)$$

We propose to change the form of the action;

$$S'_\phi = \frac{1}{2\pi\gamma^2} \int_0^t d\sigma^1 \left( \frac{d}{d\sigma^1} \log \ell(\sigma^1) \right)^2 + \frac{1}{8\pi} \sum_{n \neq 0} \int_0^t d\sigma^1 \left( \dot{\phi}_n \dot{\phi}_{-n} + (2\pi n)^2 \phi_n \phi_{-n} \right) \quad . \quad (22)$$

This is equivalent to ignoring the second part of eq. (21). Accordingly

$$Z_\phi(t) \rightarrow Z'_\phi(t) = \int [\mathcal{D} \log \ell] [\mathcal{D} \tilde{\phi}] e^{-S'_\phi} = \langle \log \ell_2; t \mid \log \ell_1 \rangle \prod_{n \neq 0} \langle 0; t \mid 0 \rangle_n \quad . \quad (23)$$

Here the first factor represents the transition amplitude of quantum mechanics of the length variable and the remaining infinite product represents the vacuum amplitude for an infinite number of oscillators. Evaluating this amplitude, we obtain

$$\mathcal{Z}_{\text{cont}} \rightarrow \mathcal{Z}'_{\text{cont}}(\ell_1, \ell_2) = \frac{2\sqrt{2}}{\gamma} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_2 / \ell_1)^2}{2\pi\gamma^2 t}} \eta(q = e^{-4\pi t}) Z_m(t) \quad . \quad (24)$$

The factor  $\frac{1}{t^{1/2}}$  comes from  $\langle \log \ell_2; t \mid \log \ell_1 \rangle$ . Comparing eq. (24) with eq. (17), we see

$$\gamma^2 = 4 \quad , \quad Z_m(t) = \eta(e^{-4\pi t})^2 \quad . \quad (25)$$

Although our discussion leading to eq. (24) is heuristic, let us take that the argument of [9] can be applied here. This will give us  $\gamma = \frac{1}{2\sqrt{3}} (\sqrt{1-c} \mp \sqrt{25-c})$  and to recover the correct semi-classical limit in the spherical topology, we must choose the minus sign in this formula. We find that  $\gamma^2 = 4$  is obtained if and only if  $c = -2$  and we choose the unconventional branch, *i.e.* the plus sign.<sup>3</sup> The expression eq. (25) for the  $Z_m(t)$  is reproduced by the path integral of the first order system in which only the nonzero oscillating modes are included. Note that our discussion essentially differs from that of [11] in the treatment of zero modes. The bosonization formula of [12] does not apply in our case.

On the other hand, in the case of  $c \rightarrow 1$  ( $m \rightarrow \infty$ ), the two-loop amplitude can be expressed as

$$\begin{aligned} w(\ell_1, \ell_2)_{c \rightarrow 1}^{M \rightarrow 0} &\equiv \lim_{m \rightarrow \infty} \lim_{M \rightarrow 0} w(\ell_1, \ell_2) \\ &= \frac{\sqrt{2\beta}}{\pi} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_1 / \ell_2)^2}{8\pi\beta t}} \\ &\quad \times \sum_{n=0}^\infty \int_0^1 d\nu (n+\nu) \sin \{(n+\nu)\pi\} e^{-2\pi\beta(n+\nu)^2 t} \\ &= \frac{1}{8\beta\pi} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\frac{(\log \ell_1 / \ell_2)^2}{8\pi\beta t}} \frac{1}{t^{3/2}} e^{-\frac{\pi}{8\beta t}} \\ &= \frac{1}{(\log \frac{\ell_2}{\ell_1})^2 + \pi^2} . \end{aligned} \quad (26)$$

$$= \frac{1}{(\log \frac{\ell_2}{\ell_1})^2 + \pi^2} . \quad (27)$$

---

<sup>3</sup>It is curious that this unconventional branch also appears in the recent discussion of touching interactions on random surfaces [10].

Note that in this case we have no criterion to fix  $\beta$ . That eq. (26) contains no Dedekind function has a well-known interpretation in the continuum framework. When the cosmological constant is vanishing, the Liouville field acts as an extra conformal matter field. Therefore the target space is two-dimensional, which does not allow a string to vibrate. The cancellation of the nonzero modes can be explicitly seen in the continuum both for the case of torus [13] and for the case of annulus [14].

We have derived the explicit expression for  $w(\ell_1, \ell_2)_c$  as well as the one at  $M \rightarrow 0$ . Our integral representation in this limit has an interpretation from the continuum path integrals for the cases of  $m = 2$  ( $c = 0$ ) and  $m = \infty$  ( $c = 1$ ). Our discussion suggests that, for the case of pure gravity, the amplitude is essentially controlled by the lowest critical point  $(2, 1)$ , where the central charge is  $c = -2$  and the only operator of the theory is the boundary operator [15]. For the case of  $c = 1$ , our result is consistent with the continuum calculation.

Finally, we present an outline of the derivation of eq. (7) and that of eq. (9) to the extent space permits. We first quote a formula of a definite integral

$$\int_0^\infty e^{-ax} K_\nu(x) = \frac{\pi}{\sin \nu\pi} \frac{\sinh[\nu \log(a + \sqrt{a^2 - 1})]}{\sqrt{a^2 - 1}}, \quad (28)$$

where  $|\operatorname{Re} \nu| < 1$ ,  $\operatorname{Re} a > -1$ . By setting  $\nu$ ,  $x$  and  $a$  to be  $\frac{k}{m}$ ,  $M\ell$  and  $\cosh m\theta$  respectively, we obtain

$$\begin{aligned} \int_0^\infty e^{-\zeta\ell} K_{\frac{k}{m}}(M\ell) M d\ell &= \frac{\pi}{\sin \frac{k\pi}{m}} \frac{\sinh k\theta}{\sinh m\theta} \\ &\equiv \mathcal{L}[MK_{\frac{k}{m}}(M\ell)]. \end{aligned} \quad (29)$$

Taking a derivate with respect to  $\zeta$ , we find

$$\frac{\pi}{\sin \frac{k\pi}{m}} \frac{\partial}{\partial \zeta} \frac{\sinh k\theta}{\sinh m\theta} = -\mathcal{L}[M\ell K_{\frac{k}{m}}(M\ell)]. \quad (30)$$

The inverse Laplace transform of this eq. provides eq. (7).

We quote another formula of an indefinite integral

$$\int^z dz z Z_\nu(\alpha z) Z_\nu^*(\beta z) = \frac{z}{\alpha^2 - \beta^2} \left( \beta Z_\nu(\alpha z) Z_{\nu-1}^*(\beta z) - \alpha Z_{\nu-1}(\alpha z) Z_\nu^*(\beta z) \right), \quad (31)$$

where  $\alpha \neq \beta$  and  $Z_\nu(z)$  and  $Z_\nu^*(z)$  represent either the Bessel function, the Neumann function or the Hankel function. If we take  $Z_\nu(z)$  and  $Z_\nu^*(z)$  to be the Hankel function  $H^{(1)}(z) = \frac{2}{i\pi} e^{-i\pi\nu/2} K_\nu(-iz)$ , we find

$$\int^z dz z K_\nu(\alpha z) K_\nu(\beta z) = \frac{z}{\beta^2 - \alpha^2} \left( \beta K_\nu(\alpha z) K_{\nu-1}(\beta z) - \alpha K_{\nu-1}(\alpha z) K_\nu(\beta z) \right). \quad (32)$$

Setting  $(z, \nu, \alpha, \beta)$  to be  $(M, \frac{k}{m}, \ell_1, \ell_2)$  and using the identity  $K_\nu = K_{-\nu}$ , we obtain the formula eq. (9).

## References

- [1] G. Moore, N. Seiberg and M. Staudacher, *Nucl. Phys.* **B362** (1991) 665.
- [2] T. Banks, M. Douglas, N. Seiberg and S. Shenker, *Phys. Lett.* **B238** (1990) 279.
- [3] L. Alvarez-Gaumé, H. Itoyama, J.L. Manes and A. Zadra, *Int. J. Mod. Phys.* **A7** (1992) 5337.
- [4] G. Moore and N. Seiberg, *Int. J. Mod. Phys.* **A7** (1992) 2601.
- [5] J.M. Daul, V.A. Kazakov and I.K. Kostov, *Nucl. Phys.* **B409** (1993) 311.
- [6] I. K. Kostov, *Nucl. Phys.* **B376** (1992) 539.
- [7] R. Nakayama, *Phys. Lett.* **B325** (1993)347.
- [8] E. D’Hoker and D. H. Phong, *Rev. Mod. Phys.* **60** (1988)917.
- [9] J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989)509; F. David, *Mod. Phys. Lett.* **A3** (1988)1651.
- [10] I. Klebanov, PUPT-1486, July 1994, hep-th 9407167.
- [11] J. Distler, *Nucl. Phys.* **B342** (1990)523.
- [12] D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986)93.
- [13] M. Bershadsky and I. R. Klebanov, *Phys. Rev. Lett.* **65** (1990)3088.
- [14] A. Ishikawa, *Phys. Rev.* **D50** (1994)2609.
- [15] E. Martinec, G. Moore and N. Seiberg, *Phys. Lett.* **B263** (1991) 190.